8th Week

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Subspaces

$S \subseteq V$, to be a subspace ;

0_V

1) $\vec{0}_V \in S$ ✓

2) $\forall \vec{s}_1, \vec{s}_2 \in S$ $\quad \vec{s}_1 \oplus \vec{s}_2 \in S$ ✓ (closed)

3) $\forall r \in \mathbb{R}$ $\quad \forall \vec{s} \in S$ $\quad r \odot \vec{s} \in S$ ✓

$\in \mathbb{R}$ vector

$\Rightarrow S \leq V$

$(V, \oplus, \odot)$

$S \subseteq V$

$S = \{ \quad : \text{conditions} \}$ such that

$S \leq V$
S is a subspace of V

Ex/ $V \to \mathbb{R}^3 \longrightarrow \{(a,b,c) : a,b,c \in \mathbb{R}\}$

n-tuples ⊆ n-tuples

$S = \{ (x,y) : \text{conditions} \} \not\subseteq V$

$\Rightarrow S \not\leq V$

$\mathbb{R}^3 = \{ (a,b,c) : a,b,c \in \mathbb{R} \}$

$\mathbb{R}^n = \{ (x_1, x_2, \dots, x_n) : x_i \in \mathbb{R} \}$

Ex/ $V = \mathbb{R}^3$

$S = \{ (x,y) : x < y \}$ $\quad$ Is S a subspace of $\mathbb{R}^3$?

2-tuples

$S \not\subseteq \mathbb{R}^3 \Rightarrow S \not\leq \mathbb{R}^3$

Ex/ $V = \mathbb{R}^3$

$S = \{ (x,y,z) : z = 0 \}$ $\quad$ Is $S \leq V$? $\quad$ $(1,2,3) \notin S$ $\quad$ $(3,5,0) \in S$

condition

1) $\vec{0}_V = (0,0,0) \overset{?}{\in} S$ ✓

2) $(x_1, y_1, z_1) \in S \Rightarrow (x_1, y_1, 0)$
$(x_2, y_2, z_2) \in S \quad\;\; (x_2, y_2, 0)$

$(x_1,y_1,0) \oplus (x_2,y_2,0) = (x_1+x_2, y_1+y_2, 0+0) \overset{?}{\in} S$ ✓

$=0$ ✓

3) $\forall r \in \mathbb{R}$ $\quad (x,y,z) \in S$, $z = 0$

worst possibility

$r \odot (x,y,0) = (rx, ry, r.0) \in S$ ✓
$=0$

$\Rightarrow S \leq V$

Ex/ $V = \mathbb{R}^2$ $\quad$ Is $S \leq \mathbb{R}^2$? $\quad$ $(5,1) \in S$ $\quad$ $(2,3) \notin S$

$S = \{ (x,y) : x \geq y \}$

$S \subseteq \mathbb{R}^2$

1) $\vec{0}_V = (0,0) \overset{?}{\in} S$ $\quad 0 \geq 0$ ✓ $\quad (0,0) \in S$ ✓

2) $(x_1, y_1) \in S \to x_1 \geq y_1$
$(x_2, y_2) \in S \to x_2 \geq y_2$

$(x_1, y_1) \oplus (x_2, y_2) = (x_1+x_2, y_1+y_2) \overset{?}{\in} S$ ✓

$\Longrightarrow x_1 + x_2 \geq y_1 + y_2$ ✓

$(x_2, y_2) \in S \to x_2 \geqslant y_2$ $\xrightarrow{\text{implies}}$ $x_1 + x_2 \geqslant y_1 + y_2$ ✓

3) $\forall r \in \mathbb{R}$ $\forall (x,y) \in S$, $x \geqslant y$ $\qquad r \odot (x,y) = (rx, ry) \overset{?}{\in} S$ ✗

$rx \overset{?}{\geqslant} ry$

a counter example: → $r = -2$ $\quad (x,y) = (4,3) \to 4 \geqslant 3$

$-2 \odot (4,3) = (-8,-6) \notin S$

$-8 \ngeqslant -6$

$\Rightarrow S \nleqslant V$

From Ch# 3.2

1. Determine whether the following sets form subspaces of $\mathbb{R}^2$:

(a) $\{(x_1, x_2)^T \mid x_1 + x_2 = 0\}$
(b) $\{(x_1, x_2)^T \mid x_1 x_2 = 0\}$
(c) $\{(x_1, x_2)^T \mid x_1 = 3x_2\}$
→ **(d)** $\{(x_1, x_2)^T \mid |x_1| = |x_2|\}$
→ **(e)** $\{(x_1, x_2)^T \mid x_1^2 = x_2^2\}$ $\quad |x_1| = |x_2|$

$(2,2) \in S$ ✓
$(-2,2) \in S$
$(3,5) \notin S$

d) $S = \{ (x_1, x_2) : |x_1| = |x_2| \}$ $\qquad S \subseteq \mathbb{R}^2$ is $S \leqslant \mathbb{R}^2$?

1) $\vec{0}_{\mathbb{R}^2} = (0,0) \overset{?}{\in} S$ ✓ since $|0| = |0|$ ✓

✗ 2) $(x_1, x_2) \in S \Rightarrow |x_1| = |x_2|$
$(y_1, y_2) \in S \Rightarrow |y_1| = |y_2|$

$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$

$|x_1 + y_1| \overset{?}{=} |x_2 + y_2|$

be the bad guy

→ $(2,2) \in S$ $\quad |2| = |2|$
→ $(-3,3) \in S$ $\quad |-3| = |3|$

$(2,2) + (-3,3) = (-1,5) \notin S$

$|-1| \neq |5|$

$\Rightarrow S \nleqslant \mathbb{R}^2$

2. Determine whether the following sets form subspaces of $\mathbb{R}^3$:

~~**(a)**~~ $\{(x_1, x_2, x_3)^T \mid x_1 + x_3 = 1\}$ $\quad 0_v \notin S$
(b) $\{(x_1, x_2, x_3)^T \mid x_1 = x_2 = x_3\}$
(c) $\{(x_1, x_2, x_3)^T \mid x_3 = x_1 + x_2\}$
→ **(d)** $\{(x_1, x_2, x_3)^T \mid x_3 = x_1 \text{ or } x_3 = x_2\}$

d) $S = \{ (x_1, x_2, x_3) : x_3 = x_1 \text{ or } x_3 = x_2 \}$

→ 1) $\vec{0}_{\mathbb{R}^3} = (0,0,0) \overset{?}{\in} S$ ✓ $\quad 0 = 0$ ✓

✗ 2) $(x_1, x_2, x_3) \in S \Rightarrow x_3 = x_1$ or $x_3 = x_2$
$(y_1, y_2, y_3) \in S \Rightarrow y_3 = y_1$ or $y_3 = y_2$

$(x_1, x_2, x_3) \oplus (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$ → is this a must?

a counter example

→ $(1,2,1) \in S$ ✓
→ $(4,3,3) \in S$ ✓

$(1,2,1) + (4,3,3) = (5,5,4) \notin S$

$\Rightarrow S \nleqslant \mathbb{R}^3$

matrix addition, scalar multiplication

3. Determine whether the following are subspaces of $\mathbb{R}^{2\times 2}$ → the vector space of all 2x2 matrices

(a) The set of all 2×2 diagonal matrices $= \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in \mathbb{R} \right\}$

(b) The set of all 2×2 triangular matrices $= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \text{either } b = 0 \text{ or } c = 0 \right\}$ (lower triangular / upper triangular)

(c) The set of all 2×2 lower triangular matrices $= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : b = 0 \right\}$

(d) The set of all 2×2 matrices A such that $a_{12} = 1$ $= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : b = 1 \right\}$ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin$

(e) The set of all 2×2 matrices B such that $b_{11} = 0$ $= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a = 0 \right\}$

(f) The set of all symmetric 2×2 matrices ($A = A^T$) $= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : b = c \right\}$

(g) The set of all singular 2×2 matrices → non-invertible $\Leftrightarrow A^{-1}$ dne $\Leftrightarrow \det(A) = 0$ $= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \underbrace{ad - bc}_{\det} = 0 \right\}$

f) $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : b = c \right\}$ Is $S \leqslant \mathbb{R}^{2\times 2}$?

1) $\vec{0}_V = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \overset{?}{\in} S$ ✓ $0 = 0$ ✓

2) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in S \Rightarrow \boxed{b = c}$

$\begin{bmatrix} d & e \\ f & g \end{bmatrix} \in S \Rightarrow \boxed{e = f}$

$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} d & e \\ f & g \end{bmatrix} = \begin{bmatrix} a+d & b+e \\ c+f & d+g \end{bmatrix} \overset{?}{\in} S$ ✓

$b + e = c + f$ implies

$b + e \overset{?}{=} c + f$ ✓

3) $\forall r \in \mathbb{R}$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in S \Rightarrow \boxed{b = c}$ $\Rightarrow r \odot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ra & rb \\ rc & rd \end{bmatrix} \overset{?}{\in} S$ ✓

$\Rightarrow rb = rc \quad \forall r \in \mathbb{R}$

$\Rightarrow S \leqslant \mathbb{R}^{2\times 2}$

g) $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad - bc = 0 \right\}$ Is $S \leqslant \mathbb{R}^{2\times 2}$?

1) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \overset{?}{\in} S$ ✓ $\det = 0 - 0 = 0$

2) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in S \Rightarrow ad - bc = 0$

$\begin{bmatrix} k & e \\ f & g \end{bmatrix} \in S \Rightarrow kg - ef = 0$

$\overset{?}{\Rightarrow} \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} k & e \\ f & g \end{bmatrix} = \begin{bmatrix} a+k & b+e \\ c+f & d+g \end{bmatrix} \overset{?}{\in} S$

det

$(a+k)(d+g) - (b+e)(c+f) \overset{?}{=} 0$

$\cancel{ad} + kd + ag + \cancel{kg} - \cancel{bc} - ec - bf - \cancel{ef} \overset{?}{=} 0$

a counter example:

$\underset{\det = 0}{\overset{\in S}{\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}}} + \underset{\det = 0}{\overset{\in S}{\begin{bmatrix} -3 & -3 \\ -3 & -3 \end{bmatrix}}} = \underset{\det \neq 0}{\overset{\notin S}{\begin{bmatrix} -1 & -5 \\ -5 & -1 \end{bmatrix}}}$

$\det = 1 - 25 = -24$ ✓

Some Special Subspaces (related to matrices)

1) Null Space of a matrix $A_{m\times n}$: $N(A) \leq \mathbb{R}^n$

$N(A)$ = the set of all solutions of $Ax = 0$

$A_{m\times n}$

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}_{m\times n} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m\times 1}$$

n-tupple

$N(A) \subseteq \mathbb{R}^n$

1) $\overbrace{(0,0,\ldots,0)}^{n} = \vec{0}_{\mathbb{R}^n} \overset{?}{\in} N(A)$ ✓ → trivial solution

2) $(x_1,x_2,\ldots,x_n) \in N(A)$ $\quad (y_1,y_2,\ldots,y_n) \in N(A)$ $\quad \underbrace{A(x+y)}_{\in N(A)} = \underbrace{Ax}_{0} + \underbrace{Ay}_{0} = 0$

$Ax=0 \qquad Ay=0$

3) $\forall r \in \mathbb{R}$ $\quad (x_1,x_2,\ldots,x_n) \in N(A) \Rightarrow Ax=0$

$r \odot (x_1,x_2,\ldots,x_n) = (rx_1, rx_2, \ldots, rx_n)$ $\quad \underbrace{A(rx)}_{\in N(A).} = r\underbrace{(Ax)}_{0} = 0$

$\Rightarrow N(A) \leq \mathbb{R}^n$

Ex/

$A = \begin{bmatrix} 1 & 2 & -3 & -1 \\ -2 & -4 & 6 & 3 \end{bmatrix}$ $\qquad N(A) = ?$ $\qquad Ax=0$

$$\left[\begin{array}{cccc|c} 1 & 2 & -3 & -1 & 0 \\ -2 & -4 & 6 & 3 & 0 \end{array}\right] \xrightarrow{2r_1+r_2\to r_2} \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & & \end{array} \\ \left[\begin{array}{cccc|c} 1 & 2 & -3 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \rightarrow x_4 = 0$$

REF ✓

$x_2 = r \in \mathbb{R}$

$x_3 = s \in \mathbb{R}$

$\Rightarrow x_1 = 3s - 2r$

$\Rightarrow N(A) = \{ (3s-2r, r, s, 0) : r, s \in \mathbb{R} \} \leq \mathbb{R}^4$

2) Row space

3) Column space

+ Span
→ spanning set